Verifica le seguenti identità:

1. $\frac{1+\tan^2\alpha}{\tan^2\alpha}+1+cot^2\left(\frac{3}{2}\pi+\alpha\right)=\frac{1}{\sin^2\alpha-\sin^4\alpha}$

$$\left(1+\frac{\sin^2\alpha}{\cos^2\alpha}\right)\cdot\frac{\cos^2\alpha}{\sin^2\alpha}+1+\tan^2\alpha=\frac{1}{\sin^2\alpha\ (1-\sin^2\alpha)}$$

$$\frac{\cos^2\alpha+\sin^2\alpha}{\cos^2\alpha}\cdot\frac{\cos^2\alpha}{\sin^2\alpha}+1+\frac{\sin^2\alpha}{\cos^2\alpha}=\frac{1}{\sin^2\alpha\cos^2\alpha}$$

$$\frac{1}{\sin^2\alpha}+1+\frac{\sin^2\alpha}{\cos^2\alpha}=\frac{1}{\sin^2\alpha\cos^2\alpha}\qquad \frac{\cos^2\alpha+\sin^2\alpha\cos^2\alpha+\sin^4\alpha}{\sin^2\alpha\cos^2\alpha}=\frac{1}{\sin^2\alpha\cos^2\alpha}$$

$$\cos^2\alpha+\sin^2\alpha\ (\cos^2\alpha+\sin^2\alpha)=1\qquad \cos^2\alpha+\sin^2\alpha=1\qquad 1=1\qquad verificata$$

2. $2\cos^2\frac{\alpha}{2}\left[1-\sin\left(\frac{\pi}{2}-\alpha\right)\right]=\frac{4\tan^2\frac{\alpha}{2}}{\left(1-\tan^2\frac{\alpha}{2}\right)^2(1+\tan^2\alpha)}$

$$2\frac{1+\cos\alpha}{2}\ (1-\cos\alpha)=\frac{4\frac{1-\cos\alpha}{1+\cos\alpha}}{\left(1-\frac{1-\cos\alpha}{1+\cos\alpha}\right)^2\left(1+\frac{\sin^2\alpha}{\cos^2\alpha}\right)}\qquad 1-\cos^2\alpha=\frac{4\frac{1-\cos\alpha}{1+\cos\alpha}}{\left(\frac{1+\cos\alpha-1+\cos\alpha}{1+\cos\alpha}\right)^2\left(\frac{\cos^2\alpha+\sin^2\alpha}{\cos^2\alpha}\right)}$$

$$1-\cos^2\alpha=4\frac{1-\cos\alpha}{1+\cos\alpha}\cdot\frac{(1+\cos\alpha)^2}{4\cos^2\alpha}\cdot\frac{\cos^2\alpha}{1}\qquad 1-\cos^2\alpha=1-\cos^2\alpha\qquad verificata$$

3. $2\cos^2(\alpha+\beta)+\sin(\pi-2\alpha)\sin 2\beta+4\sin^2\alpha\cos^2\beta=\frac{2}{1+\tan^2\beta}+\frac{2}{1+\cot^2\alpha}$

$$2\ (\cos\alpha\cos\beta-\sin\alpha\sin\beta)^2+\sin 2\alpha\sin 2\beta+4\sin^2\alpha\cos^2\beta=\frac{2}{1+\frac{\sin^2\beta}{\cos^2\beta}}+\frac{2}{1+\frac{\cos^2\alpha}{\sin^2\alpha}}$$

$$2\cos^2\alpha\cos^2\beta+2\sin^2\alpha\sin^2\beta-4\cos\alpha\cos\beta\sin\alpha\sin\beta+4\ \cos\alpha\cos\beta\sin\alpha\sin\beta+4\sin^2\alpha\cos^2\beta=$$

$$=\frac{2}{\frac{\cos^2\beta+\sin^2\beta}{\cos^2\beta}}+\frac{2}{\frac{\sin^2\alpha+\cos^2\alpha}{\sin^2\alpha}}$$

$$2\cos^2\alpha\cos^2\beta+2\sin^2\alpha\sin^2\beta+4\sin^2\alpha\cos^2\beta=2\ \cos^2\beta+2\sin^2\alpha$$

$$2\cos^2\alpha\cos^2\beta-2\ \cos^2\beta+2\sin^2\alpha\sin^2\beta-2\sin^2\alpha+4\sin^2\alpha\cos^2\beta=0$$

$$2\cos^2\beta\ (\cos^2\alpha-1)+2\sin^2\alpha\ (\sin^2\beta-1)+4\sin^2\alpha\cos^2\beta=0$$

$$-2\cos^2\beta\sin^2\alpha-2\sin^2\alpha\cos^2\beta+4\sin^2\alpha\cos^2\beta=0\qquad 0=0\qquad verificata$$

Calcola il valore delle seguenti espressioni:

4. $\sin^2\frac{\pi}{7}+\sin^2\frac{2}{7}\pi+\sin^2\frac{3}{14}\pi+\sin^2\frac{5}{14}\pi$

$$\sin^2\frac{\pi}{7}+\sin^2\frac{2}{7}\pi+\sin^2\left(\frac{\pi}{2}-\frac{2}{7}\pi\right)+\sin^2\left(\frac{\pi}{2}-\frac{\pi}{7}\right)=\sin^2\frac{\pi}{7}+\sin^2\frac{2}{7}\pi+\cos^2\frac{\pi}{7}+\cos^2\frac{2}{7}\pi=1+1=\mathbf{2}$$

5. $(\sqrt{3}+1)\left[\sin\left(\frac{2}{3}\pi-\alpha\right)-\sin\left(\frac{\pi}{6}+\alpha\right)\right]+\sqrt{2}\cos\left(\alpha-\frac{\pi}{4}\right)-2\cos\alpha$

$$(\sqrt{3}+1)\left(\frac{\sqrt{3}}{2}\cos\alpha+\frac{1}{2}\sin\alpha-\frac{1}{2}\cos\alpha-\frac{\sqrt{3}}{2}\sin\alpha\right)+\sqrt{2}\left(\frac{\sqrt{2}}{2}\cos\alpha+\frac{\sqrt{2}}{2}\sin\alpha\right)-2\cos\alpha=$$

$$=\frac{1}{2}(\sqrt{3}+1)(\sqrt{3}-1)(\cos\alpha-\sin\alpha)+\cos\alpha+\sin\alpha-2\cos\alpha=\cos\alpha-\sin\alpha-\cos\alpha+\sin\alpha=\mathbf{0}$$

6. Dimostra che in un triangolo ABC, con l'angolo $\hat{B} = 2\hat{A}$, si ha: $\frac{\tan\hat{A}\sin\hat{B}}{\sin\hat{A}} = \frac{2\sin\hat{C}}{4\cos^2\hat{A}-1}$

Poniamo $\hat{A} = \alpha$, $\hat{B} = 2\alpha$, $\hat{C} = \pi - \hat{A} - \hat{B} = \pi - \alpha - 2\alpha = \pi - 3\alpha$.

$$\frac{\tan\alpha\sin 2\alpha}{\sin\alpha} = \frac{2\sin(\pi - 3\alpha)}{4\cos^2\alpha - 1} \qquad \frac{\frac{\sin\alpha}{\cos\alpha}\cdot 2\sin\alpha\cos\alpha}{\sin\alpha} = \frac{2\sin 3\alpha}{4\cos^2\alpha - 1}$$

$$2\sin\alpha = \frac{2\sin(\alpha + 2\alpha)}{4\cos^2\alpha - 1} \qquad \sin\alpha = \frac{\sin\alpha\cos 2\alpha + \cos\alpha\sin 2\alpha}{4\cos^2\alpha - 1}$$

$$\sin\alpha = \frac{\sin\alpha\,(2\cos^2\alpha - 1) + \cos\alpha\cdot 2\sin\alpha\cos\alpha}{4\cos^2\alpha - 1}$$

$$\sin\alpha = \sin\alpha\frac{2\cos^2\alpha - 1 + 2\cos^2\alpha}{4\cos^2\alpha - 1} \qquad \sin\alpha = \sin\alpha \qquad verificata$$

7. Dato un triangolo rettangolo, indicando con α, β e γ gli angoli interni, calcola le seguenti espressioni:
   A. $\cos^2\alpha + \cos^2\beta + \cos^2\gamma$
   B. $\sin 8\alpha + \sin 8\beta + \sin 8\gamma$

Dato che α, β e γ sono gli angoli interni di un triangolo rettangolo: $\beta = \frac{\pi}{2} - \alpha$ e $\gamma = \frac{\pi}{2}$, perciò:

$$\cos^2\alpha + \cos^2\left(\frac{\pi}{2} - \alpha\right) + \cos^2\frac{\pi}{2} = \cos^2\alpha + \sin^2\alpha + 0 = \mathbf{1}$$

$$\sin 8\alpha + \sin\left[8\left(\frac{\pi}{2} - \alpha\right)\right] + \sin 8\cdot\frac{\pi}{2} = \sin 8\alpha + \sin(4\pi - 8\alpha) + \sin 4\pi = \sin 8\alpha - \sin 8\alpha + 0 = \mathbf{0}$$